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Noncommutative Schubert Calculus and Grothendieck Polynomials

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In this paper we extend the work of Fomin and Greene on noncommutative Schur functions by defining noncommutative analogs of Schubert polynomials. If the variables satisfy certain relations (essentially the same as those needed in the theory of noncommutative Schur functions), we prove a Pieri-type formula and a Cauchy identity for our noncommutative polynomials. Our results imply the conjecture of Fomin and Kirillov concerning the expansion of an arbitrary Grothendieck polynomial in the basis of Schubert polynomials; we also present a combinatorial interpretation for the coefficients of the expansion. We conclude with some open problems related to it. © 1999 Academic Press

1. INTRODUCTION

Fomin and Greene defined noncommutative Schur functions in [6] by using a certain reading of the tableaux in the combinatorial definition of Schur functions. They showed that, surprisingly, the noncommutative Schur functions commute if their variables $u_1, u_2, \ldots$ satisfy the "non-local Knuth relations"

$$\begin{aligned} u_i u_k u_j &= u_k u_i u_j, \qquad i \leqslant j < k, \qquad |i-k| \geqslant 2, \\ u_j u_i u_k &= u_j u_k u_i, \qquad i < j \leqslant k, \qquad |i-k| \geqslant 2, \end{aligned} \tag{1.1}$$

as well as the following "local commutation" relation:

$$(u_i + u_{i+1})\, u_{i+1} u_i = u_{i+1} u_i (u_i + u_{i+1}). \tag{1.2}$$

The above relations are satisfied in many well-known algebras, such as the plactic, nilplactic, nilCoxeter, and degenerate Hecke algebras. As a consequence of the above result, Fomin and Greene derived a Cauchy identity

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for the noncommutative Schur functions and a generalized Littlewood–Richardson rule for a large class of symmetric functions, including the stable Schubert and the stable Grothendieck functions.

In this paper we extend the work of Fomin and Greene to Schubert calculus. We define certain noncommutative polynomials $S_w(\mathbf{u})$, for every permutation w in the symmetric group Σ_n, by analogy with the definition of noncommutative Schur functions, and by considering the construction of Schubert polynomials due to Billey *et al.* [3] as the analog of the combinatorial definition of Schur functions. We interpret the "compatible sequences" in the above construction as certain tableaux of staircase shape with entries 0 and 1, as it is done in [1]. The noncommutative analogs of Schubert polynomials are then defined by using a certain reading of these binary tableaux. Unlike the noncommutative Schur functions, the polynomials $S_w(\mathbf{u})$ do not commute if the relations (1.1) and (1.2) are satisfied; they still do not commute if we replace the relations (1.1) with the stronger relation

$$u_i u_j = u_j u_i, \qquad |i-j| \geqslant 2. \tag{1.3}$$

Nevertheless, in Section 4 we show that our polynomials do satisfy a noncommutative version of the Cauchy identity in Schubert calculus if relations (1.3) and (1.2) are satisfied (note that (1.1) and (1.2) do not suffice). Clearly, we have to use different techniques from those in [6] to prove the Cauchy identity for $S_w(\mathbf{u})$. Indeed, the most difficult step for us is to prove a Pieri-type formula for our polynomials; this formula expresses the product of the noncommutative analog of an elementary symmetric polynomial with $S_w(\mathbf{u})$ as a sum of polynomials $S_{w'}(\mathbf{u})$, provided that $w(1)=1$. The proof of the Pieri-type formula relies heavily on a new insertion algorithm for the binary tableaux of staircase shape mentioned above. In Section 3 we concentrate on the properties of this algorithm which are needed in this paper, postponing the discussion of other properties to a future paper. Finally, in Section 5 we discuss the special case of the degenerate Hecke algebra $H_n(0)$, which is related to Grothendieck polynomials. Recall that these polynomials are representatives for the classes dual to the structure sheaves of Schubert varieties in the K-theory of the flag variety. Our Cauchy identity immediately implies a conjecture of Fomin and Kirillov concerning the expansion of a Grothendieck polynomial in the basis of Schubert polynomials; furthermore, it offers a combinatorial interpretation for the coefficients of this expansion. To be more precise, it turns out that the sign of a coefficient only depends on the degree of the corresponding Schubert polynomial. The expansion we study provides a more efficient expression for a Grothendieck polynomial than the expression in terms of its monomials. It also allows us to use properties of Schubert polynomials (which are better understood) in the study of Grothendieck polynomials. We briefly discuss the geometrical significance of the expansion mentioned above, which is still mysterious to a considerable extent. We conclude with some conjectures concerning this expansion.

2. THE DEFINITION OF THE NONCOMMUTATIVE ANALOGS OF SCHUBERT POLYNOMIALS

Recall that the nilCoxeter algebra (of the symmetric group Σ_n) is generated by elements $v_1, ..., v_{n-1}$, subject to the relations

$$\begin{aligned} v_i^2 &= 0, \\ v_i v_j &= v_j v_i, \qquad |i-j| \geqslant 2, \\ v_i v_{i+1} v_i &= v_{i+1} v_i v_{i+1}. \end{aligned} \tag{2.1}$$

The nilCoxeter algebra can also be defined as the algebra spanned by elements which can be identified with permutations in Σ_n; more precisely, the product $u_{a_1} \cdots u_{a_k}$ is identified with w in Σ_n, where $a_1 \cdots a_k$ is any reduced word for w. The multiplication is given by

$$w_1 \cdot w_2 := \begin{cases} \text{usual product } w_1 w_2 & \text{if} \quad l(w_1 w_2) = l(w_1) + l(w_2) \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

We shall not attempt to distinguish notationally between elements and their products in Σ_n and the nilCoxeter algebra, since in those cases where it matters, we have taken care to ensure that the context is clear. Note that a faithful representation of the nilCoxeter algebra is given by the well-known divided difference operators ∂_i, which we do not use in this paper, although they are commonly used to define Schubert polynomials (see Section 4).

In what follows, we will denote by $P(\cdot, v_m)$ and $P(v_m, \cdot)$ any products of the form $v_{i_1} v_{i_2} \cdots v_{i_p}$, where $i_1 > i_2 > \cdots > i_p = m$, and $v_{j_1} v_{j_2} \cdots v_{j_q}$, where $m = j_1 > j_2 > \cdots > j_q$, respectively; we use a similar notation, namely $P(\cdot, s_m)$ and $P(s_m, \cdot)$ for products of simple transpositions $s_i = (i, i+1)$ in Σ_n. If $m \leqslant 0$ or $m \geqslant n$, we set $P(\cdot, v_m)$ and $P(v_m, \cdot)$ equal to 1.

Now recall that the Schubert polynomials $\mathfrak{S}_w(\mathbf{x}) = \mathfrak{S}_w(x_1, ..., x_{n-1})$, for w in Σ_n, are polynomials of degree $l(w)$ with positive integer coefficients in the (commuting) variables $x_1, ..., x_{n-1}$. They can be defined by their generating function

$$\mathfrak{S}(\mathbf{x}) := \prod_{i=1}^{n-1} \prod_{j=n-1}^{i} (1 + x_i v_j), \qquad \text{that is,} \quad \mathfrak{S}(\mathbf{x}) = \sum_{w \in \Sigma_n} \mathfrak{S}_w(\mathbf{x}) w; \tag{2.3}$$

the variables x_i commute with v_j, and the noncommuting factors of the double product are evaluated in the specified order. This is essentially the construction of Schubert polynomials due to Billey *et al.* [3], which was reformulated and reproved by Fomin and Stanley in [8] using the nilCoxeter algebra.

The definition of Schubert polynomials in (2.3) can be reformulated in terms of certain tableaux, and thus viewed as the analog of the combinatorial definition of Schur functions. We consider the set of binary tableaux of staircase shape

$$\mathscr{T} := \{T = (t_{ij})_{1 \leqslant i \leqslant j \leqslant n-1} : t_{ij} \in \{0, 1\}\}.$$

For any T in $\mathscr{T}$ we define

$$v(T, i) := \prod_{j=n-1}^{i} v_j^{t_{ij}}, \qquad v(T) := \prod_{i=1}^{n-1} v(T, i),$$

$$x(T) := \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} x_i^{t_{ij}}.$$

For each permutation w in Σ_n, we let

$$\mathscr{T}(w) := \{T \in \mathscr{T} : v(T) = w\}.$$

Throughout this paper, a binary tableau will mean a tableau in $\mathscr{T}(w)$ for some w in Σ_n. With the above notation, we can reformulate (2.3) in the following way

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{T \in \mathscr{T}(w)} x(T). \tag{2.4}$$

We note that a similar reformulation was used in [1] to rederive some classical properties of Schubert polynomials; the corresponding binary tableaux of staircase shape, which are slightly different from the ones we use, were called RC-graphs.

Given a set of noncommuting variables $u_1, ..., u_{n-1}$ and some binary tableau T, we let

$$u(T, j) := \prod_{i=1}^{j} u_{n-i}^{1-t_{ij}}, \qquad u(T) := \prod_{j=n-1}^{1} u(T, j).$$

We now define the noncommutative analogs of Schubert polynomials $S_w(\mathbf{u})$ by

$$S_w(\mathbf{u}) = S_w(u_1, ..., u_{n-1}) := \sum_{T \in \mathscr{T}(ww_0)} u(T), \tag{2.5}$$

where $w_0 := (n, n-1, ..., 1)$ is the longest permutation in Σ_n (throughout this paper, we use the one-line notation for permutations). Similarly, we can define $S_w(u_k, ..., u_{n-1})$ for every permutation w of the set $\{k, ..., n\}$ (the corresponding symmetric group will be denoted by $\Sigma_{\{k, ..., n\}}$); in this case, we consider binary tableaux $T = (t_{ij})$ with $1 \leqslant i \leqslant n-k$ and $i+k-1 \leqslant j \leqslant n-1$, and define

$$v(T) := \prod_{i=1}^{n-k} \prod_{j=n-1}^{i+k-1} v_j^{t_{ij}}, \qquad u(T) := \prod_{j=n-1}^{k} \prod_{i=1}^{j-k+1} u_{n-i}^{1-t_{ij}}.$$

Note that

$$S_w(\mathbf{x}) = \left(\prod_{i=1}^{n-1} x_i^i \right) \mathfrak{S}_{ww_0}(x_{n-1}^{-1}, ..., x_1^{-1}).$$

Clearly, the degree of $S_w(\mathbf{u})$ is $l(w)$. For instance, if w is the identity permutation, then $S_w(\mathbf{u}) = 1$, and if $w = w_0$, then $S_w(\mathbf{u}) = (u_{n-1} \cdots u_1)(u_{n-1} \cdots u_2) \cdots u_{n-1}$. Let us consider another example.

EXAMPLE 2.6. Consider the permutation $w = (2, 3, 4, 1)$ in Σ_4, for which $ww_0 = (1, 4, 3, 2) = s_2 s_3 s_2 = s_3 s_2 s_3$. The set $\mathscr{T}(ww_0)$ consists of the following binary tableaux, with column indices decreasing from left to right.

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & \\ 0 & & \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & \\ 1 & & \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & \\ 1 & & \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & 1 & \\ 0 & & \end{matrix} \qquad \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & \\ 1 & & \end{matrix}$$

Hence $S_w(\mathbf{u}) = u_1 u_2 u_3 + u_2 u_2 u_3 + u_2 u_3 u_3 + u_3 u_1 u_3 + u_3 u_3 u_3$.

Note that the polynomials $S_w(\mathbf{u})$ are in some sense complementary to Schubert polynomials, because they are defined in terms of the entries equal to 0 of the binary tableaux T, as opposed to the entries equal to 1, which are used to define Schubert polynomials. However, this complementarity is natural to consider, as Sottile and Bergeron show in [2], where they give a formula for $x^\delta \mathfrak{S}_w(x_1^{-1}, ..., x_{n-1}^{-1})$ in terms of chains in the so-called k-Bruhat order (as usual $\delta := (n-1, n-2, ..., 1)$ and $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$). Our main reason for considering the "complementary" polynomials is that only in terms of them were we able to find a noncommutative version of the Cauchy identity for Schubert polynomials. Furthermore, our noncommutative Pieri-type formula is considerably simpler than the Pieri formula for Schubert polynomials, because it only involves the weak Bruhat order. Let us also note that neither the polynomials $S_w(\mathbf{u})$, nor their variations obtained by changing the definition of $u(T)$ (for instance, by recording the 1's rather than the 0's) are related to the noncommutative Schubert polynomials defined by Lascoux and Schützenberger in [14]. Indeed, our polynomials are not reading words of tableaux, in general. Furthermore, as we shall explain in a future paper, the polynomials $S_w(\mathbf{u})$ are related to a different phenomenon than the Edelman–Greene correspondence (see [4]), on which the construction of Lascoux and Schützenberger is based.

The polynomials $S_w(\mathbf{u})$ are stable under the obvious embedding of the symmetric group $\Sigma_{\{k+1, ..., n\}}$ into the symmetric group $\Sigma_{\{k, k+1, ..., n\}}$. To state this property, we use the standard notation $1 \times w$ for the image of a permutation $w \in \Sigma_{\{2, ..., n\}}$ in Σ_n.

PROPOSITION 2.7. *For every permutation* w *in* $\Sigma_{\{2, ..., n\}}$, *we have that* $S_w(u_2, ..., u_{n-1}) = S_{1 \times w}(\mathbf{u})$.

Proof. Let $w'_0 := (n, n-1, ..., 2)$, and $c := (2, 3, ..., n, 1)$. Then we have $(1 \times w)\, w_0 = (1 \times ww'_0)\, c$. We claim that we have a bijection $T' = (t'_{ij})_{1 \leqslant i < j \leqslant n-1} \mapsto T = (t_{ij})_{1 \leqslant i \leqslant j \leqslant n-1}$ from $\mathscr{T}(ww'_0)$ to $\mathscr{T}((1 \times w)\, w_0)$ defined by

$$t_{ij} = \begin{cases} t'_{ij} & \text{if} \quad i < j \\ 1 & \text{otherwise.} \end{cases}$$

Injectivity is clear. The only thing to check, which shows that $T \in \mathscr{T}((1 \times w)\, w_0)$, as well as surjectivity, is

$$(P(\cdot, v_2)\, v_1)(P(\cdot, v_3)\, v_2) \cdots v_{n-1} = P(\cdot, v_2)\, P(\cdot, v_3) \cdots (v_1 v_2 \cdots v_{n-1}). \tag{2.8}$$

Furthermore, we clearly have $u(T') = u(T)$, and hence the stability result. ∎

3. AN INSERTION ALGORITHM FOR BINARY TABLEAUX

In this section we present an insertion algorithm for the binary tableaux introduced in the previous section. This algorithm is our main tool for proving the noncommutative Pieri formula and Cauchy identity. Let us mention that an insertion algorithm for RC-graphs was given in [1]. The main differences between the two algorithms are: (a) we insert a 0 rather than a 1; (b) we do not stop the algorithm if at some point along the insertion path we obtain a binary tableau T with $v(T) \neq 0$; (c) the successive insertion of an increasing sequence of elements using our procedure produces paths which are weakly above one another, unlike the algorithm in [1]. The latter property is crucial in the proofs below. Note that not having property (c) prevented Bergeron and Billey from extending their proof of Monk's formula to the Pieri formula for Schubert polynomials.

ALGORITHM 3.1 (Insertion). Consider an integer i with $1 \leqslant i \leqslant n-1$ and T in $\mathcal{T}(w)$ for some w in Σ_n. Assume it is possible to find a sequence of indices (the *insertion path*) $(k_0, l_0), (k_1, l_1), ..., (k_r, l_r)$, with $1 \leqslant k_p \leqslant l_p \leqslant n-1$ for all $0 \leqslant p \leqslant r$, satisfying the following properties:

(1) $k_0 = n-i$, $l_0 = n$, and we set $t_{k_0 l_0} := 0$;

(2) if $t_{k_p l_p} = 0$ for some p with $0 \leqslant p \leqslant r-1$, then $k_{p+1} = k_p$ and $l_{p+1} = l_p - 1$;

(3) if $t_{k_p l_p} = 1$ and $t_{k_{p-1} l_{p-1}} = 0$ for some p with $1 \leqslant p \leqslant r-1$, then $k_{p+1} = k_p - 1$ and $l_{p+1} = l_p$;

(4) if $t_{k_p l_p} = 1$ and $t_{k_{p-1} l_{p-1}} = 1$ for some p with $1 \leqslant p \leqslant r-1$, then $k_{p+1} = k_p - 1$ and $l_{p+1} = l_p - 1$;

(5) $k_r = 1$ and $t_{k_r l_r} = 1$.

If such a sequence exists, it is clearly unique. We define a new binary tableau $(i \to T) = (t'_{ab})$ (the insertion of i into T) by simply setting $t'_{k_p l_p} := t_{k_{p-1} l_{p-1}}$ if $1 \leqslant p \leqslant r$, and $t'_{ab} := t_{ab}$ for all other pairs (a, b).

An example of insertion path is given in Fig. 1 (here $n = 11$ and $i = 4$). The insertion algorithm consists of shifting the entries along the insertion path; an extra 0 enters the tableau in position $(n-i, n-1)$, and the entry equal to 1 at the end of the insertion path is removed (see Fig. 2).

We now present some properties of this insertion procedure.

PROPOSITION 3.2. *Assume that in the sequence of indices* $(k_0, l_0), ..., (k_r, l_r)$ *we have a pair* (k_p, l_p) *such that* $t_{k_p l_p} = 1$ *and* $t_{k_{p-1} l_{p-1}} = 1$. *Then* $l_p - 1 \geqslant k_p$ *and* $t_{k_p, l_p - 1} = 1$.

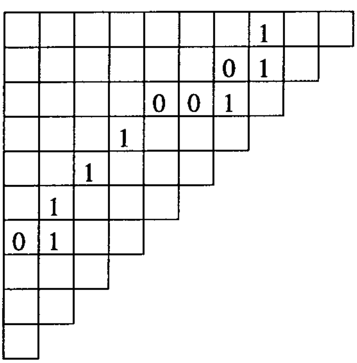

FIGURE 1

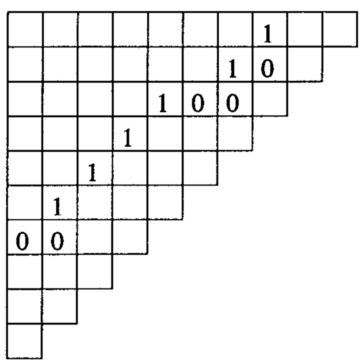

FIGURE 2

Proof. The first assertion of the proposition is clear. Assume that $t_{kl} = 0$ for some (k, l) in the insertion path, and that the following $q+1$ entries in this path are 1 ($q \geqslant 1$). According to the definition of the insertion path, the indices corresponding to the entries equal to 1 are $(k, l-1)$, $(k-1, l-1)$, $(k-2, l-2)$, ..., $(k-q, l-q)$. Assume we proved $t_{k-s,\ l-s-1} = 1$ for $s = 1, ..., q-1$, and assume for contradiction that $t_{k-q,\ l-q-1} = 0$. Then, by the relations in the nilCoxeter algebra, we can rewrite $v(T)$ in the following way:

$$\begin{aligned}
&\cdots P(\cdot, v_{l-q+1})\, v_{l-q} P(v_{l-q-2}, \cdot)\, P(\cdot, v_{l-q+2}) \\
&\qquad \times v_{l-q+1} v_{l-q} P(v_{l-q-1}, \cdot) \cdots P(\cdot, v_{l+1})\, v_{l-1} \cdots \\
&\quad = \cdots P(\cdot, v_{l-q+1})\, P(v_{l-q-2}, \cdot)\, P(\cdot, v_{l-q+2}) \\
&\qquad \times v_{l-q} v_{l-q+1} v_{l-q} P(v_{l-q-1}, \cdot) \cdots P(\cdot, v_{l+1})\, v_{l-1} \cdots \\
&\quad = \cdots P(\cdot, v_{l-q+1})\, P(v_{l-q-2}, \cdot)\, P(\cdot, v_{l-q+2}) \\
&\qquad \times v_{l-q+1} v_{l-q} v_{l-q+1} P(v_{l-q-1}, \cdot) \cdots P(\cdot, v_{l+1})\, v_{l-1} \cdots \\
&\quad = \cdots P(\cdot, v_{l-q+1})\, P(v_{l-q-2}, \cdot)\, P(\cdot, v_{l-q+2}) \\
&\qquad \times v_{l-q+1} v_{l-q} P(v_{l-q-1}, \cdot) \cdots P(\cdot, v_{l+1})\, v_{l-1} v_{l-1} \cdots.
\end{aligned}$$

Hence $v(T) = 0$, which is a contradiction with the choice of T. ∎

The above proposition simply says that, in fact, the binary tableau in Fig. 1 looks as shown in Fig. 3.

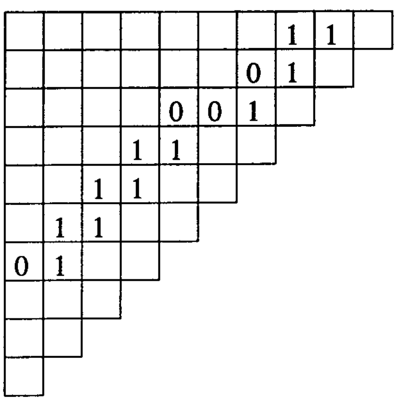

FIGURE 3

PROPOSITION 3.3. *Let T be the binary tableau considered above, and assume that the elements $u_1, ..., u_{n-1}$ satisfy* (1.3) *and*

$$u_a u_{a+1} u_a = u_{a+1} u_a u_a, \qquad u_{a+1} u_{a+1} u_a = u_{a+1} u_a u_{a+1}, \qquad 1 \leqslant a \leqslant n-2. \tag{3.4}$$

Then we have $u_i u(T) = u(i \to T)$.

Proof. Consider an integer l with $1 \leqslant l \leqslant n$, and assume there is a position p in the insertion path such that $l_p = l$, and either $p = r$ or $l_{p+1} = l-1$. Denote the elements of $i \to T$ by t'_{ab}, and set $k := k_p$, for simplicity. Also, define

$$f(k, l) := \begin{cases} n-k & \text{if} \quad t_{kl} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We will show by (decreasing) induction on l that we have

$$u_i \left(\prod_{b=n-1}^{l} u(T, b) \right) = \left(\prod_{b=n-1}^{l} u(i \to T, b) \right) u_{f(k, l)},$$

where we set $u_0 := 1$. Clearly, this holds for $l = n$ and implies $u_i u(T) = u(i \to T)$. Assume the equality holds for arbitrary $l = l_p$ such that $p < r$ (whence $l_{p+1} = l-1$), and let us prove it for $l-1$. According to the definition of the insertion path and Proposition 3.2, we have the following possible cases.

Case 1. $t_{kl} = 0$ and $t_{k, l-1} = 0$. In this case column $l-1$ of T does not change upon insertion. The induction step reduces to showing that

$$u_{n-k} u(T, l-1) = u(T, l-1)\, u_{n-k}.$$

But this follows easily by Schensted column insertion (see, e.g., [9, p. 186]).

Case 2. $k > 1$, $t_{kl} = 0$, $t_{k, l-1} = 1$, and $t_{k-1, l-1} = 0$. In this case, we have $t'_{k, l-1} = 0$ and $t'_{k-1, l-1} = 1$. The induction step reduces to

$$u_{n-k} u(T, l-1) = u(i \to T, l-1)\, u_{n-k+1},$$

which again follows by Schensted column insertion, or simply by (1.3).

Case 3. $t_{kl} = 0$, $t_{k, l-1} = 1$, and either $k = 1$ or $t_{k-1, l-1} = 1$. In this case, we have $t'_{k, l-1} = 0$ and $t'_{k-1, l-1} = 1$ (if $k > 1$). The induction step reduces to

$$u_{n-k} u(T, l-1) = u(i \to T, l-1),$$

which is straightforward by (1.3). Note that in this case we do not bump any elements from the column word $u(T, l-1)$, as Schensted insertion did if $t_{a,\,l-1}=0$ for some $1 \leqslant a \leqslant k-2$.

Case 4. $k>1$, $t_{kl}=1$, $t_{k,\,l-1}=1$, and $t_{k-1,\,l-1}=0$. In this case, we have $t'_{k,\,l-1}=1$ and $t'_{k-1,\,l-1}=1$. The induction step reduces to

$$u(T, l-1)=u(i \rightarrow T, l-1)\, u_{n-k+1},$$

which is straightforward by (1.3).

Case 5. $t_{kl}=1$, $t_{k,\,l-1}=1$, and either $k=1$ or $t_{k-1,\,l-1}=1$. In this case there is nothing to check.

Note that the use of Proposition 3.2 is crucial, since a priori we could have had $k>1$, $t_{kl}=1$, $t_{k,\,l-1}=0$, and $t_{k-1,\,l-1}=0$. ∎

In order to state the following proposition, we consider again the binary tableau T mentioned above. For every $k \leqslant n-i$, there is a unique position p in the insertion path such that $k_p=k$ and $t_{k_p l_p}=1$. We define the column index $j_k(i)$ as

$$j_k(i):=\begin{cases} l_p & \text{if } \; k_{p-1}=k \\ l_p-1 & \text{otherwise.} \end{cases} \tag{3.5}$$

Note that by Proposition 3.2 we have $t_{k,\,j_k(i)}=1$. Denote $j_1(i)$ by $j(i)$, for simplicity.

PROPOSITION 3.6. *With the notation above, we have that* $v(i \rightarrow T)=s_{j(i)}\, v(T)$, *where* $v(T)$ *and* $v(i \rightarrow T)$ *are viewed as elements of* Σ_n; *in particular,* $l(s_{j(i)}\, v(T))=l(v(T))-1$.

Proof. Throughout this proof, all products are to be evaluated in Σ_n, rather than in the nilCoxeter algebra. We use induction on $1 \leqslant k \leqslant n-i-1$ to show that

$$s_{j(i)} \left(\prod_{a=1}^{k} v(i \rightarrow T, a) \right) = \left(\prod_{a=1}^{k} v(T, a) \right) s_{j_{k+1}(i)}. \tag{3.7}$$

Then we only need to notice that $v(T, n-i)=s_{j_{n-i}(i)}\, v(i \rightarrow T, n-i)$, whence (3.7) holds for $k=n-1$ with no simple transposition in the right-hand side; since the length of the permutation in the left-hand side is at most the length of $v(T)$ (indeed, $i \rightarrow T$ has one more zero than T), we must have $\prod_{a=1}^{n-1} v(i \rightarrow T, a)=v(i \rightarrow T)$. Both the case $k=1$ and the induction step for (3.7) reduce to proving that

$$s_{j_k(i)}\, v(i \rightarrow T, k)=v(T, k)\, s_{j_{k+1}(i)}, \tag{3.8}$$

for all k with $1 \leqslant k \leqslant n-i-1$. Let $l := j_{k+1}(i)$. There are two possible cases, as follows.

Case 1. There is $q \geqslant 1$ such that $t_{kl} = \cdots = t_{k,\,l-q+1} = 0$ and $t_{k,\,l-q} = 1$. In this case, (3.8) becomes

$$s_{l-q}(P(\cdot, s_{l+1})\, s_l P(s_{l-q-1}, \cdot)) = P(\cdot, s_{l+1})\, s_{l-q} P(s_{l-q-1}, \cdot)\, s_l$$

which is obvious.

Case 2. $t_{kl} = t_{k,\,l-1} = 1$ (here we use Proposition 3.2 again). In this case, (3.8) becomes

$$s_{l-1} P(\cdot, s_{l+1})\, s_l s_{l-1} P(s_{l-2}, \cdot) = P(\cdot, s_{l+1})\, s_l s_{l-1} P(s_{l-2}, \cdot)\, s_l,$$

which is also obvious. ∎

ALGORITHM 3.9 (Reverse Insertion). Consider an integer j with $1 \leqslant j \leqslant n-1$ and T in $\mathscr{T}(w)$ for some w in Σ_n. Find the unique sequence of indices (the reverse insertion path) $(k_0, l_0), (k_1, l_1), \ldots, (k_r, l_r)$, with $1 \leqslant k_p \leqslant l_p \leqslant n-1$ for all $0 \leqslant p \leqslant r$, satisfying the following properties:

(1) $k_0 = 0$, $l_0 = j$, and we set $t_{k_0 l_0} := 1$;

(2) if $t_{k_p l_p} = 0$ for some p with $0 \leqslant p \leqslant r-1$, then $k_{p+1} = k_p$ and $l_{p+1} = l_p + 1$;

(3) if $t_{k_p l_p} = 1$ and $t_{k_p+1,\, l_p} = 0$ for some p with $1 \leqslant p \leqslant r-1$, then $k_{p+1} = k_p + 1$ and $l_{p+1} = l_p$;

(4) if $t_{k_p l_p} = 1$ and $t_{k_p+1,\, l_p} = 1$ for some p with $1 \leqslant p \leqslant r-1$, then $k_{p+1} = k_p + 1$ and $l_{p+1} = l_p + 1$;

(5) $l_r = n-1$, and $t_{k_r l_r} = 1$ implies $t_{k_r+1,\, l_r} = 1$.

Note that it is not possible to have $k_p = l_p$ and $t_{k_p l_p} = 1$, so the entry $t_{k_p+1,\, l_p}$ always exists in T if $t_{k_p l_p} = 1$. We define a new binary tableau $(T \leftarrow j) = (t'_{ab})$ (the reverse insertion of j into T) by simply setting $t'_{k_p l_p} := t_{k_{p-1} l_{p-1}}$ if $1 \leqslant p \leqslant r$, and $t'_{ab} := t_{ab}$ for all other pairs (a, b). We let $i(j) := n - k_r$.

PROPOSITION 3.10. (a) *If we can insert i into T, then $((i \to T) \leftarrow j(i)) = T$.*

(b) *If $v_j v(T) \neq 0$, then $(i(j) \to (T \leftarrow j)) = T$.*

Proof. Part (a) is immediate by Proposition 3.2 and the definitions of the two types of insertion paths above. Part (b) reduces to showing that

$t_{k_r l_r}=0$ in the reverse insertion path of j into T, as long as $v_j v(T)\neq 0$. Assume the contrary, which means that $t_{k_r+1,n-1}=1$. By (3.7), we have

$$v_j\left(\prod_{a=1}^{k_r+1} v(T,a)\right)=\left(\prod_{a=1}^{k_r} v(T\leftarrow j,a)\right) v_{n-1}v_{n-1}P(v_{n-2},\cdot)=0,$$

which leads to contradiction. ∎

We now investigate what happens if we perform a partial insertion or reverse insertion, that is, we shift the entries only along a subpath of the insertion path. First assume that $(k_0, l_0), (k_1, l_1), ..., (k_r, l_r)$ is the insertion path of i into T, and we have $k_{q-1}=k_q$, $l_{q-1}=l_q+1$, and $t_{k_q l_q}=t_{k_{q-1}l_{q-1}}=0$ for some $1\leqslant q<r$. We define a new binary tableau $T'=(t'_{ab})$ by setting $t'_{k_p l_p}:=t_{k_{p-1}l_{p-1}}$ if $1\leqslant p<q$, and $t'_{ab}:=t_{ab}$ for all other pairs (a, b). We refer to T' as the partial insertion of i into T up to position (k_q, l_q).

PROPOSITION 3.11. *Under the above assumptions, we have* $v(T')=v(T)$.

Proof. Without loss of generality, we can assume $k_q=1$. Thus, T' and $i\to T$ only differ in position $(1, l_r)$, the corresponding entries being 1 and 0, respectively. By the above assumptions, the entry in position $(1, l_r+1)$ of $i\to T$ is 0, which implies that $v(T')=s_{l_r}v(i\to T)$ (the multiplication is carried out in Σ_n). The result follows by applying Proposition 3.6. ∎

Now define a sequence $(k_1, l_1), ..., (k_r, l_r)$ as in Algorithm 3.9, with the only exception that (k_1, l_1) can be any pair of indices such that $t_{k_1 l_1}=0$. We define a new binary tableau $T'=(t'_{ab})$ by setting $t'_{k_p l_p}:=t_{k_{p-1}l_{p-1}}$ if $1<p\leqslant r$, and $t'_{ab}:=t_{ab}$ for all other pairs (a, b). We refer to T' as the reverse partial insertion for T starting at position (k_1, l_1).

PROPOSITION 3.12. *Under the above assumptions, we have* $v(T')=v(T)$.

Proof. By (2.8), we may assume, without loss of generality, that $k_1=l_1=1$. Thus, T' and $T\leftarrow 1$ only differ in position $(1, 1)$, the corresponding entries being 0 and 1, respectively. The entry in position $(1, 2)$ of $T\leftarrow 1$ is 0, which implies that $v(T\leftarrow 1)=s_1 v(T')$ (all multiplications in this proof are carried out in Σ_n). On the other hand, since $t_{11}=0$, we clearly have $v_1 v(T)\neq 0$. Hence, by Propositions 3.6 and 3.10(b), we have $v(T\leftarrow 1)=s_1 v(T)$. Finally, we obtain $v(T')=v(T)$. ∎

We are now concerned with successive insertions and reverse insertions. The following result guarantees that the corresponding insertion paths stay weakly above one another.

PROPOSITION 3.13. *Let $n-1 \geqslant i_1 \geqslant i_2 \geqslant 1$, and assume we can insert i_2 into T. If we can also insert i_1 into $i_2 \to T$, then we denote by j_1 and j_2 the corresponding column indices defined in* (3.5). *The following hold.*

(a) *If $i_1 > i_2$, then it is possible to insert i_1 into $i_2 \to T$.*

(b) *If $i_1 > i_2$, then the insertion path corresponding to i_1 stays weakly above the one corresponding to i_2. Furthermore, we have $j_1 > j_2$.*

(c) *If $i_1 = i_2 = i$ and we can insert i_1 into $i_2 \to T$, then the second insertion path stays weakly below the first one, and $j_1 \leqslant j_2$.*

Proof. (a) First note that it is possible to insert i into T if there is a strictly increasing sequence $1 \leqslant c(1) < c(2) < \cdots < c(n-i) \leqslant n-1$ such that $t_{k,\,c(k)} = 1$ for all $1 \leqslant k \leqslant n-i$; this follows easily from the definition of the insertion path. Assuming we performed the insertion $i_2 \to T$, we choose $c(1) < c(2) < \cdots < c(n-i_2-1)$ to be the columns of the 0's changed into 1's and the unchanged 1's in the corresponding insertion path. Hence it is possible to perform the insertion of i_1 into $i_2 \to T$.

(b) Denote by $(k_0^q, l_0^q), ..., (k_{r_q}^q, l_{r_q}^q)$, $q = 1, 2$, the two insertion paths. From the observation above, we can also deduce that

$$l_p^2 \leqslant c(k_p^2) \quad \text{for} \quad k_p^2 < n - i_2, \quad \text{and} \quad c(k_p^1) \leqslant l_p^1 \quad \text{for} \quad p = 0, 1, ..., r_1. \tag{3.14}$$

This means that the insertion paths stay weakly above one another. Now let us show that $j_1 > j_2$. By (3.14) we have $l_{r_1}^1 \geqslant l_{r_2}^2$, and if equality holds, then $l_{r_2}^2 = c(1)$; this implies that the insertion path of i_2 into T has only one entry (equal to 1) in the first row of T, whence $j_2 = l_{r_2}^2 - 1$. We have thus proved $j_1 \geqslant j_2$. On the other hand, we cannot have $j_1 = j_2$, by Proposition 3.6.

(c) We will show by induction on k, for $k = n-i, n-i-1, ..., 1$, that the column index of the unique entry in the second insertion path which is equal to 1 and is situated in row k is strictly smaller than the corresponding column index for the first insertion path. This is clearly true for $k = n-i$. For arbitrary k with $1 < k \leqslant n-i$, the portion of the first insertion path situated in rows $k-1$ and k can be of the following four types.

$$\begin{array}{cccccccc} & 0 \cdots 1 & & 1 & & 0 \cdots 1 & & 1 \\ 0 \cdots 1 & & 0 \cdots 1 & & 1 & & 1 & \end{array}$$

After the first insertion, the entries of these paths are the following ones.

$$\begin{array}{cccccccc} & 1 \cdots 0 & & 1 & & 1 \cdots 0 & & 1 \\ * \cdots 0 & & * \cdots 0 & & 1 & & 1 & \end{array}$$

Here $*$ stands for either 0 or 1, and the dots stand for some sequence of 0's (possibly empty). One can check without difficulty that the induction step works in each of the four cases. One useful observation is the fact that if $t_{k_p l_p}=1$ is an entry in the second insertion path (in $i \to T$), and if $t_{k_p, l_p+1}=1$, then $k_{p+1}=k_p-1$ and $l_{p+1}=l_p-1$. ∎

Now fix a permutation w in Σ_n with $w(1)=1$. According to (2.8), for every binary tableau $T=(t_{kl})$ in $\mathscr{T}(ww_0)$ we have $t_{kk}=1$ for $1 \leqslant k \leqslant n-1$. Hence, by Proposition 3.13(a) and (b), we can perform the successive insertions $(i_1 \to (i_2 \to \cdots (i_m \to T) \cdots))$, and we have $n-1 \geqslant j(i_1) > j(i_2) > \cdots > j(i_m) \geqslant 1$. Proposition 3.6 implies that $v(T_1)=s_{j(i_1)} s_{j(i_2)} \cdots s_{j(i_m)} ww_0$; in particular, we have $l(s_{j(i_1)} s_{j(i_2)} \cdots s_{j(i_m)} ww_0)=l(ww_0)-m$. Now let $\mathscr{S}_m$ denote the set of strictly decreasing sequences $n-1 \geqslant i_1 > i_2 > \cdots > i_m \geqslant 1$, and let

$$\Sigma_n^{(m)}(w') := \{w'' \colon w''=s_{j_1} \cdots s_{j_m} w', \quad n-1 \geqslant j_1 > \cdots > j_m \geqslant 1,\ l(w'')=l(w')-m\}.$$

According to the discussion above, we can define a map from $\mathscr{S}_m \times \mathscr{T}(ww_0)$ to $\bigcup_{w' \in \Sigma_n^{(m)}(ww_0)} \mathscr{T}(w')$ by the iterative insertion procedure.

PROPOSITION 3.15. *The iterative insertion procedure described above defines a bijection from the set* $\mathscr{S}_m \times \mathscr{T}(ww_0)$ *to* $\bigcup_{w' \in \Sigma_n^{(m)}(ww_0)} \mathscr{T}(w')$, *which increases the number of zeros in a binary tableau by* m.

Proof. Given a binary tableau T in $\bigcup_{w' \in \Sigma_n^{(m)}(ww_0)} \mathscr{T}(w')$, the corresponding sequence $n-1 \geqslant j_1 > \cdots > j_m \geqslant 1$ is well-defined. By Propositions 3.13(c) and 3.10, the paths corresponding to the successive reverse insertions $(\cdots((T \leftarrow j_1) \leftarrow j_2) \cdots \leftarrow j_m)$ stay weakly above one another, and we have $n-1 \geqslant i(j_1) > i(j_2) > \cdots > i(j_m) \geqslant 1$. Hence, by Proposition 3.6, the reverse insertion procedure defines a map from $\bigcup_{w' \in \Sigma_n^{(m)}(ww_0)} \mathscr{T}(w')$ to $\mathscr{S}_m \times \mathscr{T}(ww_0)$. Using Proposition 3.10 again, we can easily check that the two maps defined above are inverse to one another. ∎

4. A NONCOMMUTATIVE PIERI-TYPE FORMULA AND CAUCHY IDENTITY

Throughout this section, we assume that we are working in a noncommutative algebra containing elements $u_1, \ldots, u_{n-1}$ which satisfy relations

(1.3) and (1.2). The noncommutative analogs of the elementary symmetric polynomials $e_m(\mathbf{u})$ were defined in [6] by

$$e_m(\mathbf{u}) := \sum_{n-1 \geqslant i_1 > i_2 > \cdots > i_m \geqslant 1} u_{i_1} u_{i_2} \cdots u_{i_m}.$$

Slightly more generally, we can define $e_m(u_k, u_{k+1}, ..., u_l)$ for all $1 \leqslant k \leqslant l \leqslant n-1$ with $l-k+1 \geqslant m$. We now present a Pieri-type formula for our noncommutative Schubert polynomials, which expresses the noncommutative product $e_m(\mathbf{u}) \, S_w(\mathbf{u})$ as a sum of polynomials $S_{w'}(\mathbf{u})$, provided that $w(1) = 1$; note that a priori it is not clear that such an expression should exist. Our formula is easiest to express using the nilCoxeter algebra, and the convention $S_0(\mathbf{u}) := 0$.

THEOREM 4.1. *For every integer* m *with* $1 \leqslant m \leqslant n-1$, *and every* w *in* Σ_n *with* $w(1) = 1$, *we have that*

$$e_m(\mathbf{u}) \, S_w(\mathbf{u}) = \sum_{n-1 \geqslant j_1 > j_2 > \cdots > j_m \geqslant 1} S_{v_{j_1} v_{j_2} \cdots v_{j_m} w}(\mathbf{u}).$$

Proof. By Propositions 3.3 and 3.15, the above formula is clearly true as long as the elements u_i satisfy (1.3) and (3.4). The main point of the theorem is that we can get a stronger version of Proposition 3.3 if we consider insertions of several elements into several binary tableaux, as stated above.

Consider a binary tableau $T = (t_{ab})$ in $\mathscr{T}_{ww_0}$, and a sequence $n-1 \geqslant i_1 > i_2 > \cdots > i_m \geqslant 1$. Denote $(i_1 \to (i_2 \to \cdots (i_m \to T) \cdots))$ by $(\mathbf{i} \to T) = (\tilde{t}_{ab})$. Let $1 \leqslant k_1 < k_2 < \cdots < k_p \leqslant l+1$ be the row indices of the entries of the insertion paths corresponding to the successive insertions of $i_m, i_{m-1}, ..., i_1$, which are equal to 0 and are situated in column $l+1$; here it is understood that the entries in an insertion path are those at the moment of the corresponding insertion. Similarly, let $1 \leqslant \tilde{k}_1 < \tilde{k}_2 < \cdots < \tilde{k}_q \leqslant l$ be the corresponding row indices for column l. The strict inequalities between k_r and $\tilde{k}_r$, which were claimed above, are immediate to check. With this notation, we have

$$\begin{aligned} &u_{i_1} \cdots u_{i_m} u(T) - u(\mathbf{i} \to T) \\ &\quad = \sum_{l=n-1}^{1} \left(\prod_{b=n-1}^{l+1} u(\mathbf{i} \to T, b) \right) \\ &\qquad \times (u_{n-k_1} \cdots u_{n-k_p} u(T, l) - u(\mathbf{i} \to T, l) \, u_{n-\tilde{k}_1} \cdots u_{n-\tilde{k}_q}) \\ &\qquad \times \left(\prod_{b=l-1}^{1} u(T, b) \right). \end{aligned} \tag{4.2}$$

Now find the maximal contiguous parts in column l of T which are of the following types, where the arrows on the left and right mark the rows indexed by some k_r and some $\widetilde{k}_r$, respectively.

$$
\begin{array}{cccc}
0 & 0 \rightarrow & & 0 \rightarrow \\
0 & \rightarrow 0 \rightarrow & & 0 \rightarrow \\
\dots & \dots & & \dots \\
\rightarrow 0 \rightarrow & \rightarrow 0 \rightarrow & & 0 \rightarrow \\
0 & \rightarrow 0 \rightarrow & 1 & 0 \rightarrow \\
\rightarrow 0 \rightarrow & \rightarrow 1 & \rightarrow 1 & 1
\end{array}
$$

In the parts of the first type, the arrows can occur anywhere on the left, while the arrows on the right occur in the same positions. A part of the third type may also consist of a single row (the one with the arrow), provided that it is the first row in T. We also consider the parts of column l which lie between the parts mentioned above, and call them of the fifth type, with the remark that none of the rows k_r or $\widetilde{k}_r$ intersect these parts. One might also note that the four types of parts above correspond precisely to Cases 1–4 in the proof of Proposition 3.3. Below we show the entries occupying the same positions in the binary tableau $\mathbf{i} \rightarrow T$ as the entries shown above.

$$
\begin{array}{cccc}
0 & 1 \rightarrow & & 1 \rightarrow \\
0 & \rightarrow 0 \rightarrow & & 1 \rightarrow \\
\dots & \dots & & \dots \\
\rightarrow 0 \rightarrow & \rightarrow 0 \rightarrow & & 1 \rightarrow \\
0 & \rightarrow 0 \rightarrow & 1 & 1 \rightarrow \\
\rightarrow 0 \rightarrow & \rightarrow 0 & \rightarrow 0 & 1
\end{array}
$$

Let $a_s < c_s$ be the indices of the top and bottom rows corresponding to the parts of five types (numbered from top to bottom) into which column l was subdivided. We define

$$
u^{(s)}(T, \mathbf{k}, l) := \left(\prod_{a_s \leqslant k_r \leqslant c_s} u_{n-k_r} \right) \left(\prod_{\substack{t_{rl}=0 \\ a_s \leqslant r \leqslant c_s}} u_{n-r} \right),
$$

$$
u^{(s)}(T, l, \widetilde{\mathbf{k}}) := \left(\prod_{\substack{\widetilde{t}_{rl}=0 \\ a_s \leqslant r \leqslant c_s}} u_{n-k_r} \right) \left(\prod_{a_s \leqslant \widetilde{k}_r \leqslant c_s} u_{n-\widetilde{k}_r} \right).
$$

It is straightforward to check, using (1.3) only, that

$$u_{n-k_1}\cdots u_{n-k_p}u(T,l)=\prod_s u^{(s)}(T,\mathbf{k},l),$$

$$u(\mathbf{i}\to T,l)\,u_{n-\tilde{k}_1}\cdots u_{n-\tilde{k}_q}=\prod_s u^{(s)}(\mathbf{i}\to T,l,\tilde{\mathbf{k}}).$$

Hence, we can rewrite (4.2) in the following way:

$$\begin{aligned}
&u_{i_1}\cdots u_{i_m}u(T)-u(\mathbf{i}\to T)\\
&\quad=\sum_{l=n-1}^{1}\left(\prod_{b=n-1}^{l+1}u(\mathbf{i}\to T,b)\right)\\
&\qquad\times\left(\sum_s\left(\prod_{s'<s}u^{(s')}(\mathbf{i}\to T,l,\tilde{\mathbf{k}})\right)(u^{(s)}(T,\mathbf{k},l)-u^{(s)}(\mathbf{i}\to T,l,\tilde{\mathbf{k}}))\right.\\
&\qquad\left.\times\left(\prod_{s'>s}u^{(s')}(T,\mathbf{k},l)\right)\right)\times\left(\prod_{b=l-1}^{1}u(T,b)\right).
\end{aligned}$$

Now let us note that $u^{(s)}(T,\mathbf{k},l)=u^{(s)}(\mathbf{i}\to T,l,\tilde{\mathbf{k}})$ for all s except those corresponding to parts of column l of the first type. So fix s corresponding to a part of the first type. Assume we have

$$u^{(s)}(T,\mathbf{k},l)=(u_{n-k_\alpha}u_{n-k_{\alpha+1}}\cdots u_{n-k_\gamma})(u_{n-a_s}u_{n-(a_s+1)}\cdots u_{n-c_s}),$$

whence $u^{(s)}(\mathbf{i}\to T,l,\tilde{\mathbf{k}})$ is the same product with the two factors in opposite order. Also assume that the insertion paths containing the entries with coordinates $(k_\gamma,l),(k_{\gamma-1},l),\ldots,(k_\alpha,l)$ correspond to the insertions of i_β, $i_{\beta-1},\ldots,i_{\beta-(\gamma-\alpha)}$. Now pick rows $a_s\leqslant k'_\alpha<k'_{\alpha+1}<\cdots<k'_\gamma\leqslant c_s$. Perform the insertions of $i_m,i_{m-1},\ldots,i_{\beta+1}$ into T, then the partial insertions of $i_\beta,i_{\beta-1},\ldots,i_{\beta-(\gamma-\alpha)}$ into the resulting binary tableau up to positions $(k_\gamma,l),(k_{\gamma-1},l),\ldots,(k_\alpha,l)$ in the corresponding insertion paths; then perform the reverse partial insertions starting at $(k'_\alpha,l),(k'_{\alpha+1},l),\ldots,(k'_\gamma,l)$, and finally, the reverse insertions of $j(i_{\beta+1}),j(i_{\beta+2}),\ldots,j(i_m)$ (the latter notation was introduced in (3.5)). Denote the result of this process by T', and note that by Propositions 3.6, 3.11, and 3.12, we have $T'\in\mathcal{T}_{ww_0}$. Let $(i'_1,\ldots,i'_m)$ be the sequence obtained from $(i_1,\ldots,i_m)$ by replacing $i_{\beta-(\gamma-\alpha)},\ldots,i_\beta$ and $i_{\beta+1},\ldots,i_m$ with the rows where the corresponding reverse partial insertions, respectively reverse insertions above, end. By Proposition 3.13, the sequence $(i'_1,\ldots,i'_m)$ is strictly decreasing. By writing $u_{i'_1}\cdots u_{i'_m}u(T')-u(\mathbf{i}'\to T)$ in the form (4.3), we can see that all factors in

the term corresponding to (l, s) match, except for the difference $u^{(s)}(T, \mathbf{k}, l) - u^{(s)}(\mathbf{i} \to T, l, \tilde{\mathbf{k}})$, which becomes

$$(u_{n-k'_\alpha} \cdots u_{n-k'_\gamma})(u_{n-a_s} \cdots u_{n-c_s}) - (u_{n-a_s} \cdots u_{n-c_s})(u_{n-k'_\alpha} \cdots u_{n-k'_\gamma}).$$

It is easy to see that we have in fact defined an equivalence relation on tuples $(T; i_1, ..., i_m; l, s)$. Furthermore, the sum of terms corresponding to tuples in an equivalence class can be written as a product containing the commutator of

$$e_{\gamma-\alpha+1}(u_{n-c_s}, ..., u_{n-a_s}) \qquad \text{and} \qquad e_{c_s-a_s+1}(u_{n-c_s}, ..., u_{n-a_s})$$

as one of the factors. But it was shown in [6] that such elements commute if (1.3) and (1.2) hold. We conclude the proof by using Proposition 3.15. ▮

Now let us recall the Cauchy identity in Schubert calculus (see, e.g., [16]):

$$\prod_{i+j\leqslant n} (x_i - y_j) = \sum_{w\in\Sigma_n} \mathfrak{S}_w(\mathbf{x})\, \mathfrak{S}_{ww_0}(-\mathbf{y}). \tag{4.4}$$

The following noncommutative generalization of this identity holds. Its proof is based on the noncummutative Pieri-type formula in Theorem 4.1.

THEOREM 4.5. *Under the above assumptions, we have*

$$\prod_{i=1}^{n-1} \prod_{j=n-1}^{i} (1 + x_i u_j) = \sum_{w\in\Sigma_n} \mathfrak{S}_w(\mathbf{x})\, S_w(\mathbf{u}).$$

Proof. We use induction on n, which clearly starts at $n=1$. Assuming the identity holds for $n-1$, we have

$$\begin{aligned}
&\prod_{i=1}^{n-1} \prod_{j=n-1}^{i} (1 + x_i u_j)\\
&\quad = \left(\sum_{m=0}^{n-1} x_1^m e_m(\mathbf{u})\right)\left(\sum_{w\in\Sigma_{\{2, ..., n\}}} \mathfrak{S}_w(x_2, ..., x_{n-1})\, S_{1\times w}(\mathbf{u})\right)\\
&\quad = \sum_{w\in\Sigma_{\{2, ..., n\}}} \sum_{m=0}^{n-1} \sum_{n-1\geqslant j_1>\cdots>j_m\geqslant 1} x_1^m \mathfrak{S}_w(x_2, ..., x_{n-1})\, S_{v_{j_1}\cdots v_{j_m}(1\times w)}(\mathbf{u})\\
&\quad = \sum_{w\in\Sigma_n} \left(\sum_{m=0}^{n-1} \sum_{1\times w'\in\Sigma_n^{(m)}(w)} x_1^m \mathfrak{S}_{w'}(x_2, ..., x_{n-1})\right) S_w(\mathbf{u})\\
&\quad = \sum_{w\in\Sigma_n} \mathfrak{S}_w(\mathbf{x})\, S_w(\mathbf{u}).
\end{aligned}$$

The first equality follows by induction and the stability property in Proposition 2.7; the second equality is an application of the Pieri-type formula in Theorem 4.1; finally, the fourth equality follows from a formula in [12] for expressing a Schubert polynomial as a univariate polynomial in the first variable with coefficients being Schubert polynomials in the rest of the variables. ∎

If the u_i satisfy the relations (2.1) defining the nilCoxeter algebra, then the noncommutative Cauchy identity above implies $S_w(\mathbf{v}) = w$, according to (2.3). This means that there is a unique binary tableau T^* in $\mathscr{T}(ww_0)$ with $u(T^*) = w$, and for all the other tableaux T we have $u(T) = 0$; the latter fact can actually be proved directly without difficulty, although we do not present this proof here. Nevertheless, we identify the binary tableau T^*.

PROPOSITION 4.6. *Given w in Σ_n and assuming u_i satisfy* (2.1), *the unique binary tableau T^* in $\mathscr{T}(w)$ with $u(T^*) = ww_0$ is the maximal one in lexicographic order (here $T^* = (t^*_{ij})$ is identified with the binary word $t^*_{1,n-1} \cdots t^*_{11} t^*_{2,n-1} \cdots t^*_{22} \cdots t^*_{n-1,n-1}$). The entries of this tableau are*

$$t^*_{i,k+i-1} = \begin{cases} 1 & \text{if} \quad i \leqslant c_k(w^{-1}) \\ 0 & \text{otherwise}, \end{cases}$$

where $i, k \geqslant 1$, $i + k \leqslant n$, and $c(w^{-1}) = (c_1(w^{-1}), ..., c_{n-1}(w^{-1}))$ is the code of the inverse permutation to w.

Proof. Let us first note that if T is maximal in $\mathscr{T}(w)$ with respect to lexicographic order, then every sequence $(t_{1k}, t_{2,k+1}, ..., t_{n-k,n-1})$, for $1 \leqslant k \leqslant n-1$, is of the form $(1, 1, ..., 1, 0, 0, ..., 0)$, possibly with no 0's or no 1's at all. This is an immediate consequence of Theorem 3.7(b) in [1], which says that if T does not have the above form, then there is a lexicographically greater binary tableau in $\mathscr{T}(w)$. The fact that $v(T^*) = w$ is easiest to see if we use RC-graphs and their representation with strands, rather than our binary tableaux (see [1]); this fact is implicit in the discussion preceding Theorem 3.7 in [1]. Finally, we can prove that $u(T^*) = ww_0$ by a similar argument, using the fact that $c((ww_0)^{-1}) = -c(w^{-1})$. ∎

5. THE EXPANSION OF GROTHENDIECK POLYNOMIALS IN THE BASIS OF SCHUBERT POLYNOMIALS

We begin this section with a brief introduction to the cohomology and K-theory of flag varieties; for more information, we refer the reader to [9, 11].

Let Fl_n be the variety of complete flags $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ in $\mathbb{C}^n$; this is an irreducible algebraic variety of complex dimension $\binom{n}{2}$. Its integral cohomology ring $H^*(Fl_n)$ is isomorphic to $\mathbb{Z}[x_1, ..., x_n]/I_n$, where I_n is the ideal generated by symmetric functions in $x_1, ..., x_n$ with constant term 0; here, the elements x_i are identified with the Chern classes of the dual line bundles $(V_i/V_{i-1})^*$. Recall that Fl_n is a disjoint union of cells indexed by permutations w in Σ_n, and that their closures are the so-called Schubert varieties X_w, of complex dimension $l(w)$. It is well-known that the cohomology class corresponding to X_w is represented by the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$.

The K-theory $K^0(Fl_n)$ of the flag variety is the Grothendieck ring of complex vector bundles over Fl_n under direct sum and tensor product. A simple argument, based on the fact that the Atiyah–Hirzebruch spectral sequence collapses, shows that $K^0(Fl_n)$ is isomorphic to the same ring as $H^*(Fl_n)$. This time we identify x_i with the K-theory Chern class $1 - 1/a_i$ of the line bundle $(V_i/V_{i-1})^*$, where a_i represents V_i/V_{i-1} in the Grothendieck ring. The classes dual to the structure sheaves of Schubert varieties form the natural basis of $K^0(Fl_n)$. The construction of these classes in the general case of flag varieties corresponding to Kac–Moody Lie algebras was given in [10]; this construction is based on certain divided difference operators, as shown below. For the flag variety Fl_n, the K-theory classes corresponding to Schubert varieties are represented by Grothendieck polynomials, which were introduced by Lascoux and Schützenberger in [13], and studied in more detail in [11]. We also discuss them below.

Given a parameter β, we define polynomials $\mathfrak{G}_w^{(\beta)}(\mathbf{x}) = \mathfrak{G}_w^{(\beta)}(x_1, ..., x_{n-1})$ by

$$\mathfrak{G}_{w_0}^{(\beta)}(\mathbf{x}) := \prod_{i=1}^{n-1} x_i^{n-i},$$

$$\mathfrak{G}_w^{(\beta)}(\mathbf{x}) = \pi_i^{(\beta)} \mathfrak{G}_{ws_i}^{(\beta)}(\mathbf{x}), \qquad \text{if} \quad l(ws_i) = l(w) + 1.$$

Hence $\pi_i^{(\beta)}$ is the operator on $\mathbb{Z}[x_1, ..., x_n]$ defined by

$$\pi_i^{(\beta)} f(\mathbf{x}) = \frac{(1 + \beta x_{i+1})\, f(\mathbf{x}) - (1 + \beta x_i)\, f(x_1, ..., x_{i+1}, x_i, ..., x_n)}{x_i - x_{i+1}}.$$

The Grothendieck polynomial indexed by the permutation w is $\mathfrak{G}_w^{(-1)}(\mathbf{x})$, which we denote simply by $\mathfrak{G}_w(\mathbf{x})$. Note that $\mathfrak{G}_w^{(0)}(\mathbf{x})$ is just the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$.

It is easy to check that the operators $\pi_i^{(\beta)}$ provide a faithful representation of the algebra $\mathscr{A}_n^{(\beta)}$ generated by $u_1, ..., u_{n-1}$, subject to the relations

$$\begin{aligned} u_i^2 &= \beta u_i, \\ u_i u_j &= u_j u_i, \qquad |i-j| \geqslant 2, \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1}. \end{aligned} \tag{5.1}$$

The algebra $\mathscr{A}_n^{(\beta)}$ has a basis consisting of elements which can be identified with permutations in Σ_n in the same way as the basis elements for the nilCoxeter algebra were. Once again, we shall not attempt to distinguish notationally between elements and their products in Σ_n and $\mathscr{A}_n^{(\beta)}$. Note that $\mathscr{A}_n^{(0)}$ is the nilCoxeter algebra, and $\mathscr{A}_n^{(-1)}$ is the degenerate Hecke algebra $H_n(0)$.

Fomin and Kirillov gave a construction for the polynomials $\mathfrak{G}_w^{(\beta)}(\mathbf{x})$ similar to (2.3) in [7]. They proved that if the u_i satisfy (5.1), and $\mathfrak{G}^{(\beta)}(\mathbf{x})$ is given by the same expression as $\mathfrak{S}(\mathbf{x})$, except that the v_i's are replaced by the u_i's, then we have

$$\mathfrak{G}^{(\beta)}(\mathbf{x}) = \sum_{w \in \Sigma_n} \mathfrak{G}_w^{(\beta)}(\mathbf{x})\, w. \tag{5.2}$$

By taking a certain limit of the polynomials $\mathfrak{G}_w^{(\beta)}(\mathbf{x})$, we obtain power series in β, denoted $G_w^{(\beta)}(x_1, x_2, \ldots)$, whose coefficients are symmetric functions in $x_1, x_2, \ldots$. There is a generating function formula similar to (5.2) for these power series. We call $G_w^{(-1)}(x_1, x_2, \ldots)$ a stable Grothendieck function.

It follows from (5.2) that the Grothendieck polynomial $\mathfrak{G}_w(\mathbf{x})$ is a non-homogeneous polynomial with monomials of degree greater or equal to $l(w)$; furthermore, the sign of the coefficient of any monomial of degree $l(w)+i$ is $(-1)^i$. On the other hand, the definition of Grothendieck polynomials implies that the lowest homogeneous component of $\mathfrak{G}_w(\mathbf{x})$ is the corresponding Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$. Hence the transition matrix from Grothendieck to Schubert polynomials is triangular with 1's on the diagonal. The latter assertion has the following geometrical explanation. The fact that the Atiyah–Hirzebruch spectral sequence converging to $K^0(Fl_n)$ collapses implies that there is a descending filtration $K^0(Fl_n) \supset K^0(Fl_n)_0 \supset K^0(Fl_n)_1 \supset \cdots$ of $K^0(Fl_n)$ with $K^0(Fl_n)_k / K^0(Fl_n)_{k+1}$ isomorphic to the kth cohomology group $H^k(Fl_n)$; if $Fl_n^{(k)}$ denotes the kth skeleton of Fl_n, then $K^0(Fl_n)_k$ is just the kernel of the projection map $K^0(Fl_n) \to K^0(Fl_n^{(k)})$. In other words, the cohomology of Fl_n is the associated graded ring to $K^0(Fl_n)$ with respect to the above filtration. The same filtration of $K^0(Fl_n)$ is described in an algebraic language in [10] using the concepts of nilHecke and Hecke rings.

Fomin and Greene used their theory of noncommutative Schur functions to show that the stable Grothendieck functions are nonnegative integer combinations of Schur functions and gave a combinatorial interpretation for the coefficients of the expansion (see [6]). Here we extend their work, by providing explicit combinatorial information about the expansion of a Grothendieck polynomial in the basis of Schubert polynomials. We also confirm the conjecture of Fomin and Kirillov in [7] concerning the signs of the coefficients in this expansion.

THEOREM 5.3. *The sign of the coefficient of the Schubert polynomial $\mathfrak{S}_{w'}(\mathbf{x})$ (where $l(w') \geqslant l(w)$) in the expansion of $\mathfrak{G}_w(\mathbf{x})$ is $(-1)^{l(w')-l(w)}$. Furthermore, the absolute value of this coefficient is equal to the number of binary tableaux T in $\mathcal{T}(w'w_0)$ with $u(T)=w$, where u_i satisfy (5.1) with $\beta=1$.*

Proof. It suffices to work with the polynomials $\mathfrak{G}_w^{(1)}(\mathbf{x})$, in which all monomials have nonnegative coefficients. By (5.2), we have that $\mathfrak{G}_w^{(1)}(\mathbf{x})$ is the coefficient of w in $\mathfrak{G}^{(1)}(\mathbf{x})$. Since the relations (5.1) with $\beta=1$ are special cases of (1.3) and (1.2), we can rewrite $\mathfrak{G}^{(1)}(\mathbf{x})$ using the noncommutative Cauchy identity in Theorem 4.5. The theorem now follows by recalling the definition of the polynomials $S_w(\mathbf{u})$ in (2.5). ∎

Note that $(-1)^{l(w')-l(w)}$ is precisely the value of the Möbius function of the Bruhat order on the symmetric group. Hence it is natural to expect the following result, conjectured by Lascoux.

Conjecture 5.4. Any Schubert polynomial is a nonnegative integer combination of Grothendieck polynomials.

In [15] we proved that Conjecture 5.4 is true in the Grassmannian case, and we presented a combinatorial interpretation for the coefficients in the corresponding expansion. We state one more conjecture, which was suggested by several computer experiments.

Conjecture 5.5. We have that $\mathfrak{G}_w(\mathbf{x})=\mathfrak{S}_w(\mathbf{x})$ if and only if w is a dominant permutation.

Recall that dominant permutations are those whose code is a partition. It is known that if w is dominant, then

$$\mathfrak{G}_w(\mathbf{x})=\mathfrak{S}_w(\mathbf{x})=x_1^{c_1(w)}\cdots x_{n-1}^{c_{n-1}(w)}, \tag{5.6}$$

where $(c_1(w), ..., c_{n-1}(w))$ is the code of w.

As far as the geometric significance of the above results and conjectures is concerned, it is still mysterious to a considerable extent. The main reason for this is that the isomorphism between $K^0(Fl_n)$ and $H^*(Fl_n)$ defined above (using identification of Chern classes) is not entirely geometric. A geometrically defined isomorphism between $K^0(Fl_n) \otimes \mathbb{Q}$ and $H^*(Fl_n, \mathbb{Q})$ is the Chern character. However, the images of Schubert classes in K-theory under the Chern character are more complicated to describe then their images under the isomorphism used in this paper, although there is a connection between the two images.

Let us now consider an example to illustrate Theorem 5.3.

EXAMPLE 5.7. We have the following expansion for the Grothendieck polynomial $\mathfrak{G}_{(1,4,3,2)}(\mathbf{x})$:

$$\mathfrak{G}_{(1,4,3,2)}(\mathbf{x}) = \mathfrak{S}_{(1,4,3,2)}(\mathbf{x}) - 2\mathfrak{S}_{(2,4,3,1)}(\mathbf{x}) - \mathfrak{S}_{(3,4,1,2)}(\mathbf{x}) + \mathfrak{S}_{(3,4,2,1)}(\mathbf{x}).$$

The two binary tableaux counted by the coefficient of $\mathfrak{S}_{(2,4,3,1)}(\mathbf{x})$, and the binary tableaux counted by the coefficients of $\mathfrak{S}_{(3,4,1,2)}(\mathbf{x})$ and $\mathfrak{S}_{(3,4,2,1)}(\mathbf{x})$ are listed below, in this order.

$$\begin{array}{lll} 0 & 1 & 0 \\ 0 & 0 & \\ 1 & & \end{array} \qquad \begin{array}{lll} 0 & 0 & 0 \\ 0 & 1 & \\ 1 & & \end{array} \qquad \begin{array}{lll} 0 & 0 & 1 \\ 0 & 0 & \\ 1 & & \end{array} \qquad \begin{array}{lll} 0 & 0 & 0 \\ 0 & 0 & \\ 1 & & \end{array}$$

We conclude with a result which illustrates the potential in terms of applications of Theorem 5.3.

PROPOSITION 5.8. *The Grothendieck polynomial $\mathfrak{G}_w(\mathbf{x})$ is a linear combination of Schubert polynomials $\mathfrak{S}_{w'}(\mathbf{x})$ with $w \leqslant w'$ in Bruhat order.*

Proof. By Theorem 5.3, the proposition is equivalent to the statement $u(T) \leqslant v(T)\, w_0$ in Bruhat order, for an arbitrary binary tableaux T such that $v(T)$ is some permutation in Σ_n; here u_i satisfy (5.1) with $\beta = 1$, and the multiplication $v(T)\, w_0$ is performed in Σ_n. We prove this by induction on n, which clearly starts at 1, so let $n > 1$. Let $T' = (t_{ij})$, where $1 \leqslant i < j \leqslant n-1$. By (2.8), we have

$$u(T) = u(T')\, u_{i_1} \cdots u_{i_k} \qquad \text{and} \qquad v(T) = v(T')\, v_{j_1} \cdots v_{j_{n-k-1}}, \tag{5.9}$$

where $1 \leqslant i_1 < \cdots < i_k \leqslant n-1$ and $1 \leqslant j_1 < \cdots < j_{n-k-1} \leqslant n-1$ are sequences determined by the entries t_{ii}, $1 \leqslant i \leqslant n-1$. Note that $(j_1, ..., j_{n-k-1})$ is the sequence

$$(1, ..., n-i_k-1, \widehat{n-i_k}, n-i_k+1, ..., n-i_1-1, \widehat{n-i_1}, n-i_1+1, ..., n-1),$$

where some subsequences of consecutive integers might be empty; here $\hat{m}$ means the absence of the element m.

The proof relies on the following fact, which holds for any permutation w in Σ_n with $w(1)=1$,

$$s_{j_1}\cdots s_{j_{n-k-1}}w_0=w_0'(1,i_1+1)(i_1+1,i_2+1)\cdots(i_{k-1}+1,i_k+1);\quad (5.10)$$

here $w_0':=(1,n,n-1,...,2)=w_0s_{n-1}\cdots s_1$. This fact can be shown by a straightforward computation; more precisely, one multiplies both sides of (5.10) on the left by w_0 and examines the cycle structure of the obtained permutations, using the fact that $w(k)=l$ if and only if $(w_0ww_0)(n-k+1)=n-l+1$.

By the induction hypothesis, we have $u(T')\leqslant v(T')\,w_0'$. Let $w_1:=u(T')$ and $w_2:=v(T')\,w_0'$, for simplicity. By (5.9) and (5.10), what we have to prove (namely $u(T)\leqslant v(T)\,w_0$) is equivalent to

$$w_1u_{i_1}\cdots u_{i_k}\leqslant w_2(1,i_1+1)(i_1+1,i_2+1)\cdots(i_{k-1}+1,i_k+1),\quad (5.11)$$

where the multiplications in the left hand side are performed in $\mathscr{A}_n^{(1)}$. Note that the rule for multiplying a basis element w in $\mathscr{A}_n^{(1)}$ by u_i is that $w(i)$ and $w(i+1)$ are interchanged if $w(i)<w(i+1)$, and w is left unchanged otherwise. We prove (5.11) by induction on $k\geqslant 0$. The induction step consists of showing that if $k\geqslant 1$, $w_1'\leqslant w_2'$, $(w_1')^{-1}(1)\leqslant i_{k-1}+1\leqslant i_k$, and $w_2'(i_{k-1}+1)=1$, then

$$w_1'u_{i_k}\leqslant w_2'(i_{k-1}+1,i_k+1);\quad (5.12)$$

here we set $i_0:=0$. We use Ehresmann's criterion [5] for Bruhat order, according to which $w_1'\leqslant w_2'$ if and only if $\{w_1'(1),...,w_1'(m)\}\leqslant\{w_2'(1),...,w_2'(m)\}$ for all $m=1,...,n$, where $\{p_1<\cdots<p_m\}\leqslant\{q_1<\cdots<q_m\}$ if and only if $p_i\leqslant q_i$ for all $i=1,...,m$. It is easy to see that in order to prove (5.12), it suffices to check Ehresmann's condition for $m=i_k$. Let $\alpha:=w_1'(i_k)$, $\beta:=w_1'(i_k+1)$, $\gamma:=w_2'(i_k+1)$, $A:=\{w_1'(1),...,w_1'(i_k-1)\}\setminus\{1\}$, and $B:=\{w_2'(1),...,w_2'(i_k)\}\setminus\{1\}$. Ehresmann's criterion for the induction hypothesis $w_1'\leqslant w_2'$ and $m=i_k+1$ translates into

$$A\cup\{1,\alpha,\beta\}\leqslant B\cup\{1,\gamma\}.\quad (5.13)$$

There are two cases to consider: $\alpha=1$ and $\alpha>1$. In the first case, (5.13) implies $A\cup\{\beta\}\leqslant B\cup\{\gamma\}$, which proves (5.12). In the second case, (5.13) implies $A\cup\{\alpha,\beta\}\leqslant B\cup\{\gamma\}$; hence we have $A\cup\{1,\max(\alpha,\beta)\}\leqslant B\cup\{\gamma\}$, which proves (5.12) in this case. ∎

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